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Fractional quantum Hall states on a square lattice

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Abstract. We calculate the ground state energies and wavefunctions for small systems of fermions on a lattice using singly and doubly periodic boundary conditions. We find minima in the ground state energy when the magnetic flux per plaquette corresponds to filling fractions with odd denominators. We interpret these minima in terms of fractional quantum Hall states. The pair correlation function at nearest and next-nearest neighbours is calculated as a function of the repulsive interaction strength between particles.

1. Introduction

The properties of tight-binding electrons on a square lattice in a magnetic field are complicated and rich in structure. The single-particle spectrum shows such interesting features as self-similarity, nesting properties and a Landau-level-like structure which has been studied by Hofstadter [1] and later by others [2].

The problem of interacting electrons in a square lattice has attracted attention as a possible model for high-temperature superconductors. Theoretical interest has been concentrated on commensurate flux states in the t–J model [3, 4], the study of anyons on a lattice [5] and the relation to the fractional quantum Hall effect (FQHE) [6].

Recently, Canright *et al* [14] studied small systems of anyons on a lattice. They found that as a function of statistics and applied magnetic field the ground state of 'free' anyons could be either superconducting or quantum-Hall-like. Anyons are equivalent to bosons with flux tubes attached. The Aharonov–Bohm phases associated with encirclement of a flux tube reproduce the phase factor required by the statistics. Canright *et al* [14] also found that a mean field theory, which accounted for the statistics on average by treating the flux tubes attached to the particles as a homogeneous background magnetic field, predicted the right ground state for a large range of parameters.

The most widely accepted theory of the FQHE is based on Laughlin's wavefunction [7] for filling factors $1/q$ with q an odd integer and associated hierarchical models [8, 9] for other fractions with odd denominators. The Laughlin states describe incompressible fluids, as the 'binding' of all flux quanta to the positions of particles implicit in Laughlin's wavefunction is possible only at filling fractions $\nu = 1/q$. Any deviation from this filling fraction requires the nucleation of excitations.

The 'binding' of q flux quanta to each particle gives rise to q zeros in the wavefunction as a function of the relative coordinate of any pair of particles. This reduces

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the probability that any two particles approach each other closely and the interaction energy for any short-range repulsive interaction between particles is therefore lowered [10].

This interpretation of Laughlin's wavefunction emphasizes the treatment given to configurations in which pairs of particles approach each other closely. However, the success of many other descriptions of the fractional quantum Hall effect which incorporate Laughlin's 'binding' of flux quanta to particles' positions—theories based on an order parameter [11], a semiclassical theory (cooperative ring exchange) [12] and the fermion–flux composite picture [13]—shows that the effect is more robust than the interpretation outlined above might at first suggest.

The importance of zeros in the wavefunction as a function of the relative coordinate of any pair of particles to the short-range behaviour is less obvious for particles on a lattice, whereas all the 'coarse-grained' properties identified as important in the other theories should be well reproduced. We have therefore studied numerically systems of particles on a lattice and looked for fractional quantum-Hall-like minima in the ground state energy as a function of filling factor.

Particles in a magnetic filed on a lattice do not have a simple Landau level structure but have instead the Hofstadter single-particle spectrum. It is therefore not always sensible to say that all 'particles are in the lowest Landau level' as Hofstadter bands corresponding to a particular Landau level cannot always be identified. However, there is the advantage that even if the correspondence between Landau levels and Hofstadter bands is not always clear any calculations automatically take account of the inter-Landau-level transitions which are usually neglected in calculations for particles in the continuum.

2. The model

We study numerically the ground state energy of small systems of spin-polarized electrons on a square lattice coupled to a magnetic field. We identify fractional quantum Hall states on a lattice from the minima which appear in the contribution to the ground state energy from the interaction between particles as a function of filling factor. We assume that such minima become cusps in the thermodynamic limit.

We work with a tight-binding Hamiltonian describing electrons on a two-dimensional square lattice with nearest-neighbour repulsive interactions in the presence of a magnetic field:

$$\mathcal{H} = -t \sum_{\langle ij \rangle} \mathrm{e}^{\mathrm{i}\theta_{ij}} c_i^{+} c_j + HC + U \sum_{\langle ij \rangle} n_i n_j \tag{1}$$

where c_i are the usual fermion operators on the lattice, $n_i = c_i^{+} c_i$ is the particle number operator and $\langle ij \rangle$ denotes that the sums are over nearest neighbours of the particular lattice. The parameter U gives the strength of nearest-neighbour repulsion ($U > 0$). The phase factor

$$\theta_{ij} = 2\pi \int_i^j \boldsymbol{A} \cdot \mathrm{d}\boldsymbol{l} \tag{2}$$

is defined on each link so that the magnetic flux through a plaquette is given by $\Phi = (1/2\pi) \sum_{plaq} \theta_{ij}$ in units of magnetic flux quantum $\Phi_0 \equiv hc/e$. $\boldsymbol{A}$ is the vector potential.

We apply periodic boundary conditions in both directions (DPCs), and singly periodic conditions (SPCs) in the x-direction keeping hard walls in y-direction. SPCs have the advantage of allowing every value of flux per plaquette, Φ, but the hard walls may give rise to additional finite-size problems. DPCs do not have this problem but the allowed values of flux per plaquette are constrained by the requirement that the total number of flux quanta threading the system is integer. The phases θ_{ij} are distributed along the links so that the total flux through each plaquette is the same, modulo an integer.

We have diagonalized the Hamiltonian given by equation (1) numerically using the Lanczos algorithm [18] defined by the iterations:

$$\mathcal{H}|u_n\rangle = v_{n-1}|u_{n-1}\rangle + \varepsilon_n|u_n\rangle + v_n|u_{n+1}\rangle \tag{3}$$

$$\varepsilon \equiv \langle u_n|\mathcal{H}|u_n\rangle \qquad v_n \equiv \langle u_n|\mathcal{H}|u_{n+1}\rangle \tag{4}$$

with $v_{-1} = 0$, so that an orthonormal basis $\{|u_n\rangle\}$ is generated after a suitable choice of the initial state $|u_0\rangle$. Diagonalizing the tridiagonal matrix

$$\mathcal{H}_{mn} = \begin{cases} \varepsilon_m & \text{if } n = m \\ v_{m+1} & \text{if } n = m+1 \\ v_m & \text{if } n = m-1 \\ 0 & \text{otherwise} \end{cases} \tag{5}$$

we obtain the eigenvalues and eigenstates expressed in terms of $|u_n\rangle$. We have studied systems with up to five electrons on a $3\times 4, 3\times 5, 4\times 4$ and 4×5 lattices (where '4×5' means four sites in y-direction and five sites in x-direction). In all cases, less than 50 Lanczos steps were necessary to obtain the ground state with enough accuracy (one part in $\sim 10^4$). Independent runs have been carried out starting from a trial state corresponding to different configurations in order to be sure we have obtained the true ground state.

The single-particle spectrum of (1) for the non-interacting case has subbands with well-defined gaps. When the magnetic flux per plaquette, Φ, is a rational number, p/q, the spectrum has q subbands. The ground state energy for N noninteracting particles is just the sum of the N lowest single-particle (kinetic) energies which can be calculated exactly. It has an absolute minimum as a function of total flux when there is exactly one flux quantum per particle.

The ground state energy of the interacting system on a lattice varies as a function of the magnetic field as a result of the variation of the kinetic energy as well as the interaction energy. The variation in the kinetic energy reflects the variation of the (Hofstadter) band structure with flux per plaquette. For $U/t < 1$ this is the dominant effect and by itself can give rise to deep minima at certain filling fractions as has already been observed when the flux per plaquette $\Phi = \nu_l M_1 + M_2/M_1$, where ν_l is the number of electrons per site and M_1 and M_2 are integers [15]. However, this is not a feature of the continuum model in which the variation of the kinetic energy is only via ω_c, the cyclotron frequency, which is a smooth function of the magnetic field.

Our purpose is to study whether in a magnetic field particles on a lattice can reduce the interaction energy between them in a way similar to that observed in the continuum. We therefore define the quantity:

$$E_{\text{int}} = E_{\text{gs}} - \langle\Psi_{\text{gs}}|\mathcal{H}_{\text{kin}}|\Psi_{\text{gs}}\rangle \tag{6}$$

where E_{gs} is the ground state energy and $\mathcal{H}_{kin}$ is the hopping part of the Hamiltonian. The variation of E_{gs} due to the Hofstadter band structure has been removed in E_{int} making the variation of the contributions from the interaction energy more apparent.

We define the filling factor, ν, as the number of particles per plaquette divided by the flux per plaquette. For DPCs this is the same as the number of particles per site divided by the flux per plaquette. For SPCs the number of plaquettes and the number of sites is different (there are more sites than plaquettes). This is analogous to what happens in the continuum case. For DPCs [16] the total flux, N_Φ, is just given by $N_\Phi = N_e\nu^{-1}$ whereas with open boundary conditions on the disk $N_\Phi = N_e\nu^{-1} + X(\nu)$ with $X(\nu)$ a correction factor [17]. Canright *et al* [14] took ν to be the ratio of the number of electrons per site divided by the number of flux quanta per plaquette. We believe that our definition is more appropriate as it allows better comparison between systems with different sizes.

3. Results

In all cases we study, we find that for $U \gtrsim 3t$ the interaction energy, E_{int}, has deep minima at *fractional fillings with odd denominators*. As an example we plot in figure 1 the ground state energy as a function of filling fraction for a system of four electrons on a 4×4 lattice ($4/4 \times 4$) taking $U = 5t$ and assuming DPCs. Two deep minima can be seen at $\nu = 4/7$ and $\nu = 4/9$ which we identify with fractional quantum Hall states.

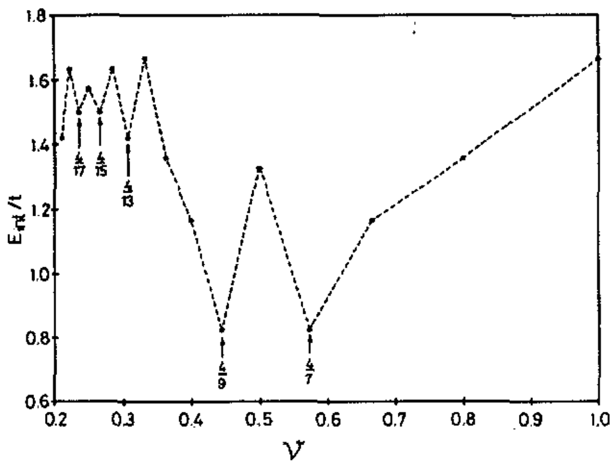

Figure 1. Ground state energy on a $4/4 \times 4$ lattice as a function of filling factor ν assuming DPCs. We identify the two deep minima with FQH states. $U = 5t$.

Comparing the results obtained for the two different types of boundary conditions we find that the results for a system with kN_e electrons using SPCs correspond well with those for a system with N_e electrons using DPCs, where the factor k is the ratio of the total number of plaquettes in the two respective cases. This implies that to compare systems with the different boundary conditions at any particular filling fraction the equivalence is set by the number of plaquettes rather than the number of sites, and is the main motivation for defining the filling fraction as we have. For a system with 4×4 sites $k = 12/16$ so that the system with SPCs corresponding to the one studied for figure 1 has three particles. The ground state energy as a function of filling fraction for three particles on a 4×4 lattice with $U = 5t$ are shown in figure 2. Deep minima are observed at the same filling fractions as those found in the equivalent system with DPCs. (There are also local minima at $\nu = 3/11$ and $\nu = 3/13$, which appear to correspond to the minima which we have observed at $\nu = 4/15$ and $\nu = 4/17$ in the corresponding $4/4 \times 4$ system using DPCs (see figure 1)).

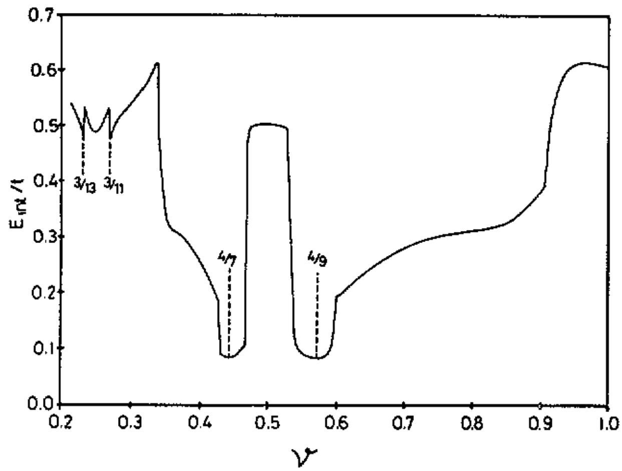

Figure 2. Ground state energy on a $3/4 \times 4$ lattices as a function of filling factor ν assuming SPCs. $U = 5t$.

In table 1 we show the filling fractions for which minima are observed for systems with 4×5 sites and three, four and five particles. We find that only filling fractions with odd denominators give minima. We also note the correspondence between systems with SPCs and DPCs. The ratio of the number of plaquettes in the two cases is 3:4 so that according to our earlier observation the system with three particles in SPCs should correspond well with that with four in DPCs. In both cases there is a deep minimum at 4/9. The deep minimum at 4/11 (four particles, DPCs) would then correspond to that at 1/3 (three particles, SPCs).

Table 1. Quantum Hall states identified from the minima in the ground state energy on an $N/4 \times 5$ lattice. The stength of interaction has been taken as $U \geqslant 4t$. Deep minima are indicated by an asterisk.

N	ν (DPCs)	ν (SPCs)
3	$\frac{3}{7}^*, \frac{1}{3}, \frac{3}{11}, \frac{3}{13}^*$	$\frac{4}{9}^*, \frac{1}{3}^*, \frac{3}{13}, \frac{1}{5}$
4	$\frac{4}{9}^*, \frac{4}{11}^*, \frac{4}{17}, \frac{1}{5}$	$\frac{4}{7}^*, \frac{4}{9}^*, \frac{5}{17}, \frac{4}{15}$
5	$\frac{5}{9}^*, \frac{5}{11}, \frac{5}{17}$	$\frac{4}{5}, \frac{5}{7}^*, \frac{5}{9}^*, \frac{2}{5}$

As mentioned in the introduction an important feature of Laughlin's wavefunction is considered to be 'binding' of zeros to the position of particles. This manifests itself in the power of r with which the pair correlation function, $g(r)$, vanishes as $r \to 0$.

To investigate short-range correlations on the lattice in the presence of magnetic field we introduce the function

$$h_\nu(R) = \frac{C_\nu(R)}{C_\infty(R)} \tag{7}$$

where

$$C_\nu(R) = N_s^{-1} \langle \Psi_{gs} | \sum_{ij} n_i n_j | \Psi_{gs} \rangle \tag{8}$$

and the sum is restricted to pairs of sites separated by a distance R. $h_\nu(R)$ measures the factor by which the probability that any two sites a distance R apart are occupied in the ground state differs when the magnetic field giving a filling factor ν is applied. $R = 1$ corresponds to nearest neighbours and $R = \sqrt{2}$ to next-nearest neighbours.

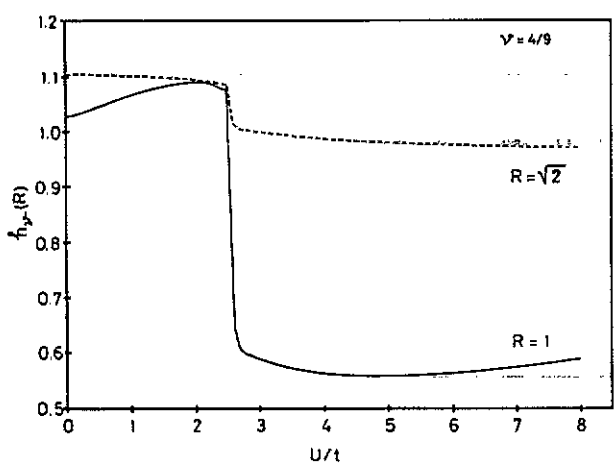

Figure 3. Nearest-neighbour ($R = 1$) and next-nearest-neighbour ($R = \sqrt{2}$) correlation function $h_\nu(R)$ for $\nu = 4/9$ as a function of the interaction strength U/t.

In figure 3 we show how for the system studied for figure 1 $h_\nu(R)$ varies with U/t when the applied magnetic field corresponds to a filling fraction 4/9 for the two cases $R = 1$ and $R = \sqrt{2}$. For $U \gtrsim 3t$ there is a significant reduction of the nearest-neighbour correlation function over its zero-field value. This occurs for a large range of values U/t and extends well beyond the case where $U = 8t$—the bandwidth. We observe similar behaviour using SPCs.

4. Discussion

We interpret our observation that deep minima in the interaction energy, E_{int} (see equation (6)) occur only at filling fractions with odd denominators as clear evidence that fractional quantum Hall states occur in small systems on a lattice.

Unfortunately we have not managed yet to resolve precisely how the various minima we observe scale with particle number, lattice size and aspect ratio. However, we have found that the results for two systems with the same flux density and lattice, one with singly periodic conditions and the other with doubly periodic boundary conditions, closely resemble each other when the number of particles per plaquette is the same in the two cases. This suggests that it is the number of plaquettes which determines the commensuration between flux and particles on a lattice.

It is our aim to study how systems of particles on a lattice behave in a magnetic field without total spin polarization. In particular we would like to see whether this can lead to states with even-denominator filling fractions.

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